Optimal control in a noisy system

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We describe a simple method to control a known unstable periodic orbit (UPO) in the presence of noise. The strategy is based on regarding the control method as an optimization problem, which allows us to calculate a control matrix **A**. We illustrate the idea with the Rossler system, the Lorenz system, and a hyperchaotic system that has two exponents with positive real parts. Initially, a UPO and the corresponding control matrix are found in the absence of noise in these systems. It is shown that the strategy is useful even if noise is added as control is applied. For low noise, it is enough to find a control matrix such that the maximum Lyapunov exponent $\lambda_{max}<0$, and with a single non-null entry. If noise is increased, however, this is not the case, and the full control matrix **A** may be required to keep the UPO under control. Besides the Lyapunov spectrum, a characterization of the control strategies is given in terms of the average distance to the UPO and the control effort required to keep the orbit under control. Finally, particular attention is given to the problem of handling noise, which can affect considerably the estimation of the UPO itself and its exponents, and a cleaning strategy based on singular value decomposition was developed. This strategy gives a consistent manner to approach noisy systems, and may be easily adapted as a parametric control strategy, and to experimental situations, where noise is unavoidable. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2956981]

Controlling chaotic systems has a range of well known techniques and procedures mainly based on the ideas of Ott, Grebogi, and Yorke for maps and in the Pyragas procedure for continuous systems, both of which use small perturbations to drive the system dynamics to a stabilized behavior. In spite of the many successes in the area, this is still a very active field of research. In this article, we explore an optimal procedure to control continuous noisy chaotic systems. Noise is unavoidable in actual experimental situations, affecting both the determination of an unstable periodic orbit (UPO) to control, and the success of a control strategy once the UPO is known. In this article, we propose a strategy to deal with both problems, thus it can be useful in a wide range of experimental situations. Besides, a given control strategy may be satisfactory in one sense, and not in others. For instance, it can keep the orbit very close to the UPO, but it may require a large effort, which can make it hard to realize experimentally. We consider this issue as well, and describe how to obtain a control matrix, optimal with respect to various criteria. This is interesting in itself, because once the problem is posed as an optimization one, as we have done in this paper, the control strategy can be further improved by using more advanced global optimization techniques.

I. INTRODUCTION

It is well known that a large number of natural[1–3] and technological[4–7] systems behave chaotically for some ranges of their parameters. In many applications chaotic behavior is undesirable, and thus regions of the parameter space where nonlinear effects are present are avoided, or the chaotic motion is eradicated by some large modification of the underlying system. Evidently, major modifications are costly and truncation of the parameter space may be too restrictive an approach. An alternative is to take advantage of basic properties inherent in a chaotic system: Unlike a linear system, it possesses many usable periodic orbits, many different time evolutions are simultaneously possible, and the motion on the chaotic attractor is exponentially sensitive to small perturbations. Ott, Grebogi, and Yorke (OGY)[8] illustrated not only that chaotic systems described by maps may be controlled, but that the richness of possible behaviors in chaotic systems may be exploited to enhance the performance of a dynamical system in a manner that would not be possible had the system's evolution not been chaotic. Shortly thereafter, Ditto *et al.*[9] reported a successful laboratory implementation of the control strategy outlined in Ref. 8, demonstrating that controlling chaos is not just a theory, but is physically attainable as well. Pyragas[10] developed these ideas in continuous dynamical systems using a delayed feedback control strategy for unstable periodic orbit (UPO).[11] This approach uses a feedback mechanism to achieve control and shares the good feature that a small perturbation is required to keep the orbit close to the desired UPO, consistent with the inherent noise level. Many studies have continued along these lines (Refs. 7 and 12–18, to cite a few). For our purpose, it is worth mentioning the study of the effect of an external noise on the controlled system,[19] and the search for optimal control strategies using a periodic driving,[20] but in which the control strength does not remain small.

In this paper, we continue along the lines set forth by Pyragas,[10] being interested in control strategies for noisy systems, where control is both optimal under a given criterion and remains small, consistent with the inherent noise level.

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We begin by describing a fairly simple method to estimate the Lyapunov spectrum of a known UPO. This method will allow us to map the issue of controlling the UPO to an optimization problem, as suggested in Refs. 18 and 20. We will use a driving term that is natural for these types of systems, as suggested by Pyragas,[10] and that can converge to a small control effort, consistent with a given noise level. Under noisy conditions, it is relevant to find optimal control strategies that minimize the effect of noise on the orbit close to the UPO. However, "optimal" does not have a unique meaning. A given strategy may be very efficient at keeping the orbit close to the UPO, but the effort to implement it can be large, a relevant issue in actual experimental situations. We take this into account in our discussion, and notice that not all criteria are equivalent. Given a noise level, we show how optimization methods allow us to find an optimal (by a certain criterion, say, the effort to maintain control) control strategy, and how the resulting control matrix depends on the level of noise. It turns out that, for low noise levels, it is enough that the control matrix has a single non-null entry, but high noise levels may require the use of the full control matrix. In either case, though, the fact is that an optimal control matrix can be found, so that control is achieved even for rather large noise levels. We illustrate these ideas with the Lorenz and Rossler systems that have a single Lyapunov exponent with a positive real part, and with a hyperchaotic system that has two Lyapunov exponents with positive real parts.

The above strategy, though, depends on the knowledge of the UPO to control, and of the associated Lyapunov exponents. However, noise can affect considerably the estimation of the exponents themselves, and the strength of the force required to keep the orbit close to the UPO. Moreover, for noisy systems, finding the UPO experimentally may be a nontrivial issue in itself. Special attention is given to this issue as well. We develop a cleaning strategy based on singular value decomposition,[21] which we illustrate with the Rossler system with added inherent noise. In particular, we show how to estimate a cleaned UPO, then compute the Lyapunov exponents, implement the optimization routine, and finally construct an optimal control matrix.

We believe that, given that noise is unavoidable in actual experimental situations, the issues discussed in this paper may be relevant to a wide range of laboratory and simulated systems.

II. CONTROL METHOD

Let us consider a dynamical nonlinear system described by

$$\dot{\vec{x}} = \mathbf{f}(\vec{x}), \tag{1}$$

where $\vec{x}$ is a vector in $\mathbb{R}^d$ and $\mathbf{f}:\mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear and at least C^1 function. Following the OGY ideas, we will take advantage of the existence of UPOs concomitant with the chaotic trajectory in the phase space.[8] Due to the ergodicity assumption, the trajectory will approach these UPOs, as close as necessary, but consistent with the noise level, as the system evolves. For this paper, controlling a UPO of period τ in this system implies (a) reducing the asymptotic average distance to the UPO for a given τ, and (b) converging to an asymptotically small average control effort, consistent with a given noise level. These two concepts will be defined in more detail later on.

For the purpose of illustration, in this work we will apply a control strategy when the state of our chaotic system is near a known target UPO $\vec{x}^*(t)$ (this is not a restriction of the method, as will be discussed below). Here $\vec{x}^*(t) \to \vec{x}^*(t \bmod \tau)$ is understood. Following Ref. 18, we can define the instantaneous distance of a trajectory to the UPO as

$$D^2(\vec{x},t) = |\vec{x} - \vec{x}^*(t)|^2$$

and we take the control as $(1/2)\mathbf{A}\vec{\nabla}_x D^2 \sim \mathbf{A}(\vec{x}-\vec{x}^*(t))$, using an optimization analogy.[21] Hence, for this work we will assume a feedback control scheme,

$$\dot{\vec{x}} = \mathbf{f}(\vec{x}) + \mathbf{A}(\vec{x} - \vec{x}^*(t)), \tag{2}$$

where the control is taken as $\vec{C}(\vec{x},t) = \mathbf{A}(\vec{x}-\vec{x}^*(t))$. Let us note that this is the form suggested by Pyragas.[10] We can control the UPO $\vec{x}^*(t)$ by finding the conditions on the matrix $\mathbf{A}$ that force a negative real part for all the exponents of the UPO, i.e., $\mathrm{Re}(\lambda_i) < 0$. In this paper, we are interested in finding the restrictions on $\mathbf{A}$ that guarantee the convergence to the UPO within a given noise level.

Of course, in some applications the control strategy is applied only when the trajectory is close to the UPO, which can be defined by a formal restriction on the values of $|\vec{C}(\vec{x},t)|$ or $D^2(\vec{x},t)$. The last restriction is necessary to ensure convergence to $\vec{x}^*(t)$ and not to other spurious UPO, as we will see below. For the rest of the paper, we will relax these restrictions and allow the control to be always active, since the time it takes for the system to reach a particular neighborhood of the UPO, so that control can be turned on, has been studied in detail in the literature. We will concentrate on characterizing (a) the Lyapunov exponents, (b) the average distance to the UPO, and (c) the effort required to keep the system under control, consistent with a given noise level.

We first turn to the estimation of the Lyapunov exponents of the UPO. In an infinitesimal neighborhood of the UPO, we have the trajectory $\vec{x}(t) = \vec{x}^*(t) + \vec{\eta}(t)$, where $\vec{\eta}$ is a small perturbation whose evolution can be approximated by

$$\dot{\vec{\eta}} = (D\mathbf{f}(\vec{x}^*) + \mathbf{A})\,\vec{\eta}, \tag{3}$$

where $D\mathbf{f}$ is the Jacobian of $\mathbf{f}$. In the Floquet approach, there exists a set of vectors $\{\vec{\eta}_1(t), \ldots, \vec{\eta}_d(t)\}$ such that

$$\vec{\eta}_i(t+\tau) = e^{\lambda_i \tau}\vec{\eta}_i(t), \tag{4}$$

where λ_i are the Floquet exponents corresponding to the $\vec{\eta}_i$ direction. We now describe a numerical method to compute the spectrum λ_i using the linearity of the above problem. There exists a fundamental matrix $\mathbf{B}(t)$ such that any solution $\vec{v}(t)$ of Eq. (3) is

$$\vec{v}(t) = \mathbf{B}(t)\vec{v}(0) \tag{5}$$

for any initial vector $\vec{v}(0)$, which requires that $\mathbf{B}(0) = \mathbf{1}$. Now let us take an arbitrary initial basis $\{\vec{v}_1(0), \vec{v}_2(0), \ldots, \vec{v}_d(0)\}$, so that an arbitrary vector $\vec{\eta}(0)$ can be written as $\vec{\eta}(0)$

$=\sum_{j=1}^{d} c_j \vec{v}_j(0)$, where the set $\vec{c}=\{c_1,\ldots,c_d\}$ satisfies $\sum_{j=1}^{d} c_j=1$. Hence we need to determine the set of coefficients that corresponds to $\vec{\eta}_i$. The above expression implies that

$$\vec{\eta}(\tau)=\mathbf{B}(\tau)\sum_{j=1}^{d} c_j\vec{v}_j(0)=\sum_{j=1}^{d} c_j\vec{v}_j(\tau), \tag{6}$$

where $\vec{v}_j(t)$ can be found numerically by integrating Eq. (3) from the initial condition $\vec{v}_j(0)$, and τ is the period of the UPO. The above relations can be rewritten in matrix form,

$$[\mathbf{V}(\tau)\mathbf{V}(0)^{-1}-e^{\lambda\tau}\mathbf{1}]\vec{c}=0, \tag{7}$$

where $\mathbf{V}(t)$ is the known matrix $\{\vec{v}_1(t),\ldots,\vec{v}_d(t)\}$. Notice that $\mathbf{B}(t)=\mathbf{V}(t)\mathbf{V}^{-1}(0)$. Solving the eigensystem (7) yields all the $\vec{\eta}_i$ vectors and their corresponding λ_i exponents.

There are a few interesting comments we can make here.

- First, we do not need to resort to infinitesimal perturbations since the above analysis can be applied equally well to small finite perturbations, in which we integrate an initial condition $\vec{y}_i(0)$ close to the UPO with Eq. (2), instead of using Eq. (3). In this case the vector for Eq. (7) is $\vec{v}_i(t)=\vec{y}_i(t)-\vec{x}^*(t)$. This situation is especially suitable to high noise levels and experimental situations, as simulated below.
- Second, in a similar manner we can apply the above method to estimate the exponents in a parametric control scheme with the dynamics given by $\dot{\vec{x}}=\mathbf{f}[\vec{x},\vec{p}^*+\mathbf{K}(\vec{x}-\vec{x}^*)]$, where $\vec{p}$ is a vector of system parameters, and $\vec{p}^*$ is their target value. In this case $\mathbf{A}\to\mathbf{K}\partial\mathbf{f}/\partial\vec{p}$, where $\mathbf{K}$ is the matrix that needs to be determined.
- Third, this methodology can be applied to experimental situations since we are only required to know the state $\vec{x}(\tau)$ after an initial condition $\vec{x}(0)$ that started close to the unstable trajectory, which can in principle be obtained as the system evolves in real time.
- Fourth, the system state can be reconstructed, if necessary, from a single or multiple time series measurements, e.g., by an appropriate embedding.[22,23]

These issues will be analyzed in detail elsewhere. For now, we turn to optimization methods to determine the matrix $\mathbf{A}$ in a feedback control strategy.

III. THE OPTIMAL MATRIX A

The Rossler system consists of three coupled ordinary differential equations[24] defined by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\begin{bmatrix}-x_2(t)-x_3(t)\\ x_1(t)+\beta_1x_2(t)\\ \beta_2+x_1(t)x_3(t)-\beta_3x_3(t)\end{bmatrix}+\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\end{bmatrix}. \tag{8}$$

We take as parameters the values $\beta_1=0.2$, $\beta_2=0.2$, and $\beta_3=4.5$, in order to ensure a bounded chaotic behavior, as shown in Fig. 1(a). From now on, a vector $\vec{x}$ represents all the dynamical variables of the system, and x_i represents its i^{th} component. The variables $\xi_i(t)$ represent a particular realization of a noise sequence, and we will describe below in detail how it is introduced in the equations.

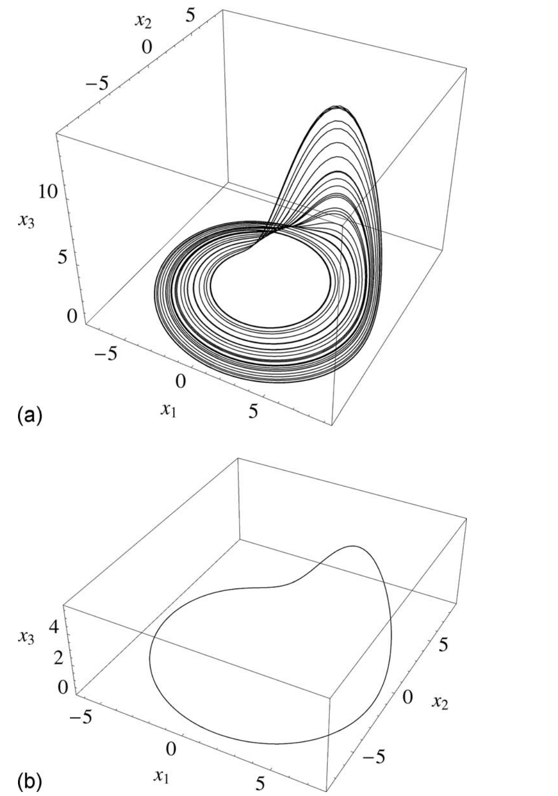

FIG. 1. (a) Rossler attractor for the parameters $\beta_1=0.2$, $\beta_2=0.2$, and $\beta_3=4.5$, showing chaotic behavior. (b) A UPO in the Rossler system with $\tau=5.86$.

Let us assume first that $\xi_i(t)=0$. Then we need to find a UPO to stabilize, represented by a local minimum of the function

$$H(\vec{x}_0,\tau)=|\vec{x}(\tau)-\vec{x}_0|^2, \tag{9}$$

where $\vec{x}(\tau)$ is the numerical solution of Eq. (8) with $\vec{x}(0)=\vec{x}_0$. Of course, we could have defined the UPO as the zero of $\vec{x}(\tau)-\vec{x}(0)$. But, let us note that to the dynamical variable $\vec{x}_0$, in d dimensions, we have added an extra variable to the function $H(\vec{x}_0,\tau)$ to optimize, namely τ, and hence a search for a zero of $\vec{x}(\tau)-\vec{x}(0)$ is not trivially defined. Of course, in the presence of noise one cannot guarantee that a minimum of $H(\vec{x}_0,\tau)$ implies a UPO of the original system (e.g., see the appearance of spurious attractors in the Lorenz system below), but if we use enough seeds in our optimization routines, and characterize the solutions with the asymptotic average distance to the UPO and the asymptotic average control effort defined below, we will eventually find UPOs.

The optimization is done with respect to τ and $\vec{x}_0$ using standard conjugate gradient or Monte Carlo methods,[21] from which we found a local minimum $\tau_0=5.86$, $x_{1,0}=0.913$, $x_{2,0}=-7.05$, $x_{3,0}=0.0418$. The corresponding UPO is shown in Fig. 1(b). We note that in general, the problem of finding a minimum of $H(\vec{x}_0,\tau)$ is a nontrivial but interesting task. There are other methods to estimate UPOs, and more advanced optimization techniques, such as genetic algorithms[25] or configurational space annealing,[26] can be used.

Initially, we will assume the usual control strategy in which the matrix $\mathbf{A}$ has a single non-null entry $A_{11}=\alpha$. The

Lyapunov exponents can be estimated from Eq. (3), where for simplicity we take as initial basis the vectors $\vec{v}_1(0)=(1,0,0)$, $\vec{v}_2(0)=(0,1,0)$, and $\vec{v}_3(0)=(0,0,1)$. Before continuing, a description of the numerical procedure is due here. A standard Runge–Kutta integrator[21] was used to numerically solve Eqs. (8) and (9) with a time step $\Delta t=0.01$ [those equations contain information equivalent to Eqs. (2) and (3) when applied to this system]. If we set the control parameter $\alpha=0$ (no control), we find for the Lyapunov exponents $\lambda_1 \approx 0.111+0.538i$, $\lambda_2 \approx -4.80\times 10^{-8}$, and $\lambda_3 \approx -2.41$. Note that as is characteristic for the UPOs in these systems, one exponent is nearly zero, one has a positive real part, and one has a negative real part. When applying the control strategy ($\alpha \neq 0$), we are interested in the range of values of α such that the real parts of all eigenvalues are negative. The maximum exponent $\lambda_{\max}$, defined as the maximum of the real parts of all exponents, can be readily calculated as a function of α, which is shown in Fig. 2(a). This figure is similar to the one estimated by Pyragas,[10] but using a different Lyapunov exponent estimation methodology.

We can now determine the value $\alpha_{\min}$ for which the maximum exponent $\lambda_{\max}$ is minimized. For the Rossler system, this occurs at $\alpha_{\min}=-1.11$ with $\lambda_{\max}=-0.38$, and the stabilized UPO, obtained by integrating the controlled Eq. (2), is shown in Fig. 2(b), after the transient has been removed. We will see below that $\alpha_0=-0.35$, corresponding to the condition $\lambda_{\max}=0$, is also a point of relevance.

Notice that, in order to be useful, the above approach must be robust in the presence of noise. The noise in the system, as described by the variables $\xi_i(t)$ in the above equations, is introduced by adding different white noise sequences $\xi_n \in [-0.1,0.1]$, where the time is discretized as $t_n = n\Delta t$, to each component of Eq. (8). For each time integration step, we assume that ξ_n is constant over the time interval $\Delta t = 1/\omega$. The same procedure will be repeated for the other systems analyzed in this paper. The white noise sequence mentioned above has standard deviation $\sigma=0.1/\sqrt{3}$, but we may also use different values of σ to study the effect of increasing noise.

In certain applications we may not be interested in $\lambda_{\max}$, but instead in minimizing the average distance to the UPO,

$$\langle d^2\rangle = \left\langle \frac{1}{T}\int_0^T |\vec{x}(t)-\vec{x}^*(t)|^2 dt \right\rangle_{\vec{x}_0},$$

or the amount of effort required to keep the orbit under control,

$$\langle F^2\rangle = \left\langle \frac{1}{T}\int_0^T |\mathbf{A}(\vec{x}(t)-\vec{x}^*(t))|^2 dt \right\rangle_{\vec{x}_0}.$$

The average is over time, over initial conditions, and over realizations of the noise sequences (ensemble average). For $\sigma=0.1/\sqrt{3}$ [Fig. 2(c)] and $\sigma=0.5/\sqrt{3}$ [Fig. 2(d)], taking $\omega = 100$ Hz, we show these two characterizations as a function of α for the Rossler system using an ensemble of initial conditions around the UPO. We note that for $\sigma=0.1/\sqrt{3}$ the curve of $\langle F^2\rangle$ has a very well defined minimum that is close to the $\lambda_{\max}=0$ case, which is represented by the vertical dashed line. This is an interesting result, as it suggests that in these systems it may be more relevant to find α_0 than $\alpha_{\min}$, for small noise levels. We will see below that this also occurs in the Lorenz system. Making the Lyapunov exponents as small as possible is a tempting criterion for "optimal" control, but in an actual experimental setup, it can be more relevant to optimize the effort to maintain control. Figures 2(a) and 2(c) show that they are not equivalent requirements.

Also, let us point out that $\langle d^2\rangle$ and $\langle F^2\rangle$ are not the only sensible choices. Other useful characterizations could be the average, in the same sense as above, of the work-like quantities $\vec{C}\cdot(\vec{x}-\vec{x}^*)$ and $\vec{C}\cdot\vec{x}$.

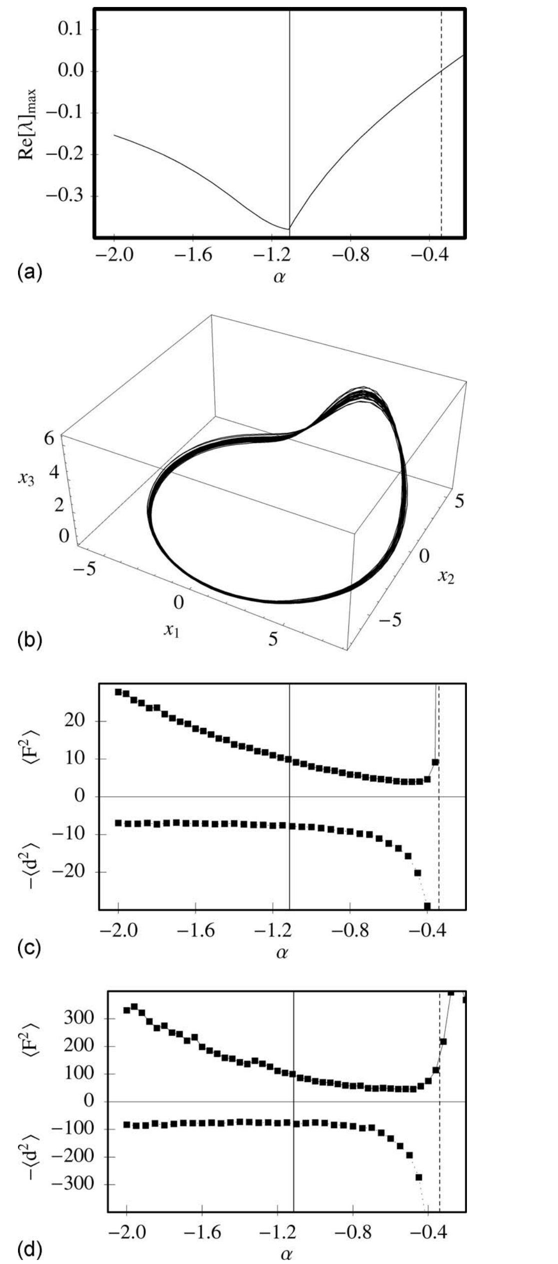

FIG. 2. (a) $\lambda_{\max}$ as a function of α for the Rossler system. The two vertical lines correspond to α_0 (dashed line) and $\alpha_{\min}$ (solid line). (b) Stabilized UPO for the Rossler system with $\alpha_{\min}=-1.11$, $\sigma=0.1/\sqrt{3}$, and $\omega=100$ Hz. We also show $-\langle d^2\rangle$ and $\langle F^2\rangle$ as a function of α for two different noise levels (c) $\sigma=0.1/\sqrt{3}$ and (d) $\sigma=0.5/\sqrt{3}$. We averaged over 50 different noise realizations, or trajectories, for 10 periods of the UPO. The transient has been removed.

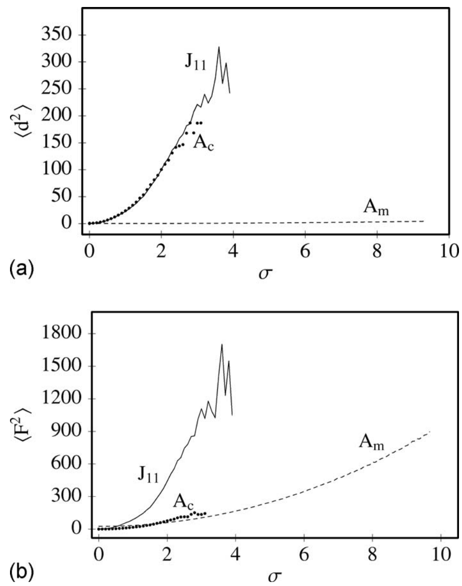

FIG. 3. (a) The average distance $\langle d^2 \rangle$ of the trajectories to the UPO, and (b) the average effort $\langle F^2 \rangle$ to keep the system under control, as a function of σ for the matrices $\mathbf{A}_c$ (dashed line), $\mathbf{A}_m$ (dotted line), and the single non-null entry matrix (continuous line) using ω=100 Hz. The cases $\mathbf{A}_c$ and $\mathbf{A}_m$ are explained below. Eventually, as we increase the noise level, the contracting effect of the controlled dynamics is not capable of compensating the expanding effect of the noise. Hence, for larger noise levels, two of the strategies are not capable of controlling the system. The close to quadratic dependence with σ is expected from the analysis described above and in the Appendix.

To understand the effect of noise, we can suggest the following argument. In a given direction, the random term is equivalent to a random walk that on average tries to push the trajectory away from the UPO by an amount $\langle \Delta x^2 \rangle \sim \sigma^2(\omega \tau)$, where σ is the standard deviation of the noise added to the system. On the other hand, the control pushes the trajectories back to the UPO as $\Delta x \sim \Delta x_0 \exp[\lambda_{\max} \tau]$ along the direction of slowest contraction, assuming $\lambda_{\max} < 0$. The two effects balance in a nontrivial manner in these systems, since the simple analysis described in the Appendix shows that the noise makes the different directions interact, i.e., the problem becomes a tensorial one, as expected for dynamical noise. Still, we see from Fig. 3 and the Appendix that the quadratic dependence with σ, derived from this simple argument, is approximately correct.

We can also observe from Figs. 2(c) and 2(d) that the value of $\langle F^2 \rangle$ increases in magnitude as the noise increases, and at the same time its minimum moves to the left, i.e., to more negative values of α. In fact, values close to α_0 are no longer capable of controlling the system, or require large amounts of control effort. This analysis suggests that as we increase the noise level, it may become more convenient to use values of α that are closer to $\alpha_{\min}$ as a benchmark. Figure 2(d) shows that $\langle d^2 \rangle$ converges for values close to $\alpha_{\min}$, even though $\langle F^2 \rangle$ is not a minimum at $\alpha_{\min}$. If we had a system and a control strategy that are required to work in a variety of noisy conditions, the choice $\alpha_{\min}$, if it exists, may be the reasonable approach. Furthermore, it suggests that in order to control UPOs under even larger noise levels, we may need to resort to the full matrix $\mathbf{A}$. We will confront this issue below.

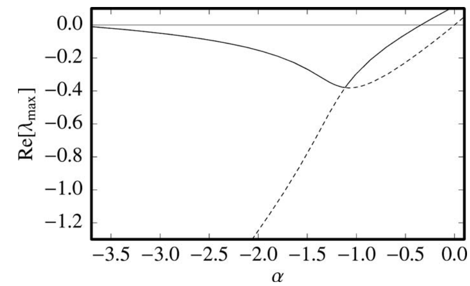

FIG. 4. Exchange between two Lyapunov exponents for the Rossler system estimated with the method described above.

Incidentally, for this UPO in the Rossler system, the optimal value of α seems to be associated with the crossing of the real part of two Lyapunov exponents as shown in Fig. 4. We notice that a strength of the above procedure for estimating the spectrum is that we can follow each exponent as a function of the control parameter in a simple manner, providing a robust way to estimate the Lyapunov exponents.

Another system of interest is the Lorenz system, described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\gamma_1 x_1(t) + \gamma_1 x_2(t) \\ \gamma_2 x_1(t) - x_2(t) - x_1(t) x_3(t) \\ x_1(t) x_2(t) - \gamma_3 x_3(t) \end{bmatrix} + \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{bmatrix}.$$

The system is chaotic for the values γ_1=16, γ_2=45.92, and γ_3=4. The variables $\xi_i(t)$ represent a particular realization of a noise sequence as described above. Let us assume first that $\xi_i(t)$=0. As done before, we find a UPO by minimizing $H(\vec{x}_0, \tau)$, from which we obtain τ=0.941, $x_{1,0}$=7.19, $x_{2,0}$ =12.4, and $x_{3,0}$=23.0. Again, we assume a control strategy in which the matrix $\mathbf{A}$ has a single non-null entry $A_{11}=\alpha$. The maximum Lyapunov exponent as a function of α is shown in Fig. 5(a). The stabilized UPO for α=−23 with $\lambda_{\max}$=−0.8 is shown in Fig. 5(b).

It is interesting to note from Fig. 5(a) that $\lambda_{\max}$ does not seem to have an optimal value, as it decreases monotonically with α. Figures 5(c) and 5(d) display $\langle F^2 \rangle$ and $-\langle d^2 \rangle$ as a function of α for two different noise levels. For $\sigma=1/\sqrt{3}$ again we see that $\langle F^2 \rangle$ has a minimum close α_0, but for large values of σ a more negative value of α is required to achieve control. Let us note that the value α=−23 provides a specific tradeoff between the optimal values of $\langle d^2 \rangle$ and $\langle F^2 \rangle$. Let us also note that while $\langle d^2 \rangle$ saturates as we make α more negative, the value $\langle F^2 \rangle$ increases as expected. Of course, in experimental situations the particular tradeoff between $\langle d^2 \rangle$ and $\langle F^2 \rangle$ would depend on the situation at hand.

Figure 5(d) also shows another interesting effect: for large values of the noise, e.g., $\sigma=5/\sqrt{3}$, the controlled trajectory may converge to some spurious attractors. These are not UPO in the native system, as far as we could resolve, but have a similar period τ to the desired trajectory, and arise due to a nontrivial interaction between the native system and the control procedure. This is an important issue to keep in mind when controlling these systems, especially in the presence of noise.

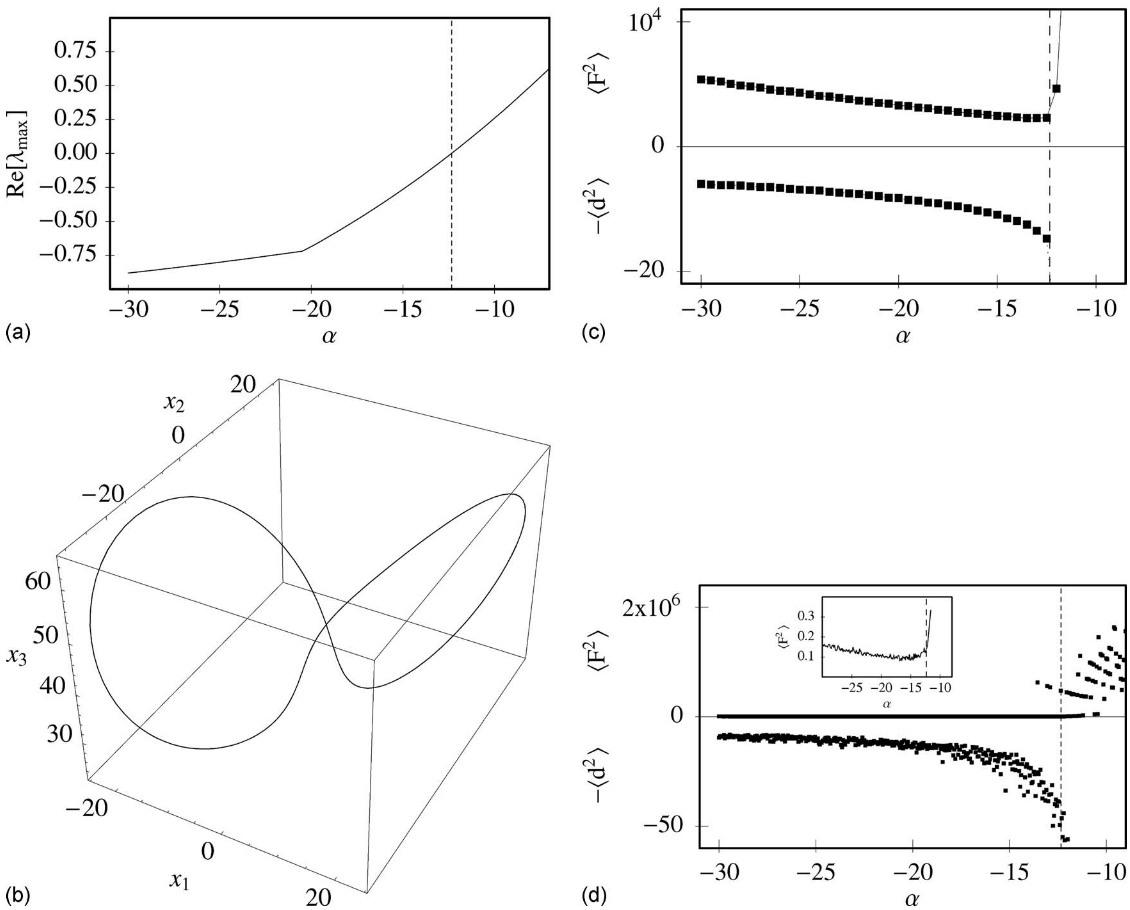

FIG. 5. (a) λ_{max} as a function of α for the Lorenz system. (b) Stabilized UPO for the Lorenz system with $\alpha=-23$, $\sigma=1/\sqrt{3}$, and $\omega=100$ Hz. We also show $-\langle d^2\rangle$ and $\langle F^2\rangle$ as a function of α for two different noise levels (c) $\sigma=1/\sqrt{3}$ and (d) $\sigma=5/\sqrt{3}$. The spurious attractors are clearly seen in $\langle d^2\rangle$ and $\langle F^2\rangle$, for the larger values of α. We average over 50 different realizations or trajectories, for 10 periods of the UPO. The transient has been removed. The inset in (d) is a zoom of the region close to the minimum of $\langle F^2\rangle$. Note that the positive and negative axes have different scales.

For instance, some authors have proposed that control could be achieved by replacing the knowledge of the UPO in the control signal, as specified by $\vec{x}^*(t)$, by a delayed state vector $\vec{x}(t-\tau)$.[10] This is an interesting idea, which may reduce the complexity in an experimental realization, but the strategy may converge to some spurious UPOs, as this work has shown. Hence, especially in the presence of noise, it may not be possible to distinguish a UPO induced by the specific control strategy from a native one, based only on the knowledge of the period τ. We may think of certain schemes that may be able to distinguish between native and spurious UPOs, but they will certainly increase the complexity of the method. In any case, these periodic orbits induced by the control strategy are something that deserves some attention, and will be explored elsewhere.

A. Optimization

It becomes clear from this analysis that the control strategy depends on what we are interested in optimizing, e.g., λ_{max}, $\langle F^2\rangle$, $\langle d^2\rangle$, etc. Besides, as the noise of the system increases, the control strategy based on the matrix with the single non-null element $A_{11}=\alpha$ may not be sufficient, or efficient enough, to keep the system under control. Indeed, there have been suggestions that a full control matrix may be the answer in systems with noise.[27,28] The required effort $\langle F^2\rangle$ as a function of the noise level σ for this strategy, with α_{min}, is shown in Fig. 3 as the continuous line, where it is compared to two other strategies explained below. As seen in the figure, $\langle F^2\rangle$ for this control strategy rapidly increases as the noise level is increased.

Therefore, it is natural to explore what happens when we relax the restriction of a single element matrix for **A**. For example, let us take a two-element matrix with $A_{11}\neq 0$ and $A_{22}\neq 0$, and arbitrarily set $\xi_i(t)=0$ to estimate λ_{max} as a function of A_{11} and A_{22}. This is shown in Fig. 6. We observe that by including the second element in the matrix, we can reduce considerably λ_{max}. Figures 6(b) and 6(c) show $\langle F^2\rangle$ for $\sigma=0.1/\sqrt{3}$ and $\sigma=0.5/\sqrt{3}$, respectively. We can clearly see how the values of the matrix elements that allow for optimal control effort change as the noise level is increased. A similar analysis can be conducted for $\langle d^2\rangle$. It becomes clear that the behavior of $\langle F^2\rangle$ is not trivial in the presence of noise, as we allow for these types of control strategies.

We will now concentrate on the problem of finding a control strategy that uses the full matrix **A**. Again, we arbitrarily set $\xi_i(t)=0$ and chose the optimization problem of finding the minimum of the scalar function λ_{max}, but now in a parameter space of dimension $d\times d$ (the number of ele-

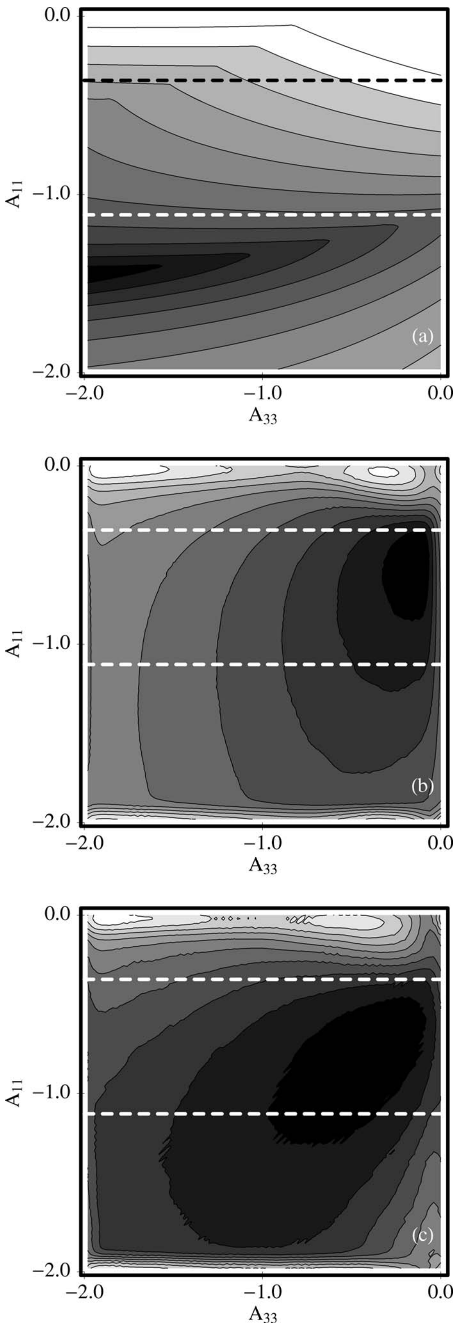

FIG. 6. (a) λ_{max} when the system is controlled by the two-element matrix ($A_{11}\neq 0$ and $A_{22}\neq 0$). The range between the smallest value of λ_{max} (−0.65) and $\lambda_{max}=0$ is divided into 10 contours such that darker means more negative values, and lighter means values closer to zero. The horizontal lines correspond to α_0 (upper line) and α_{min} (lower line). Similarly, we display $\langle F^2\rangle$ as a function of A_{11} and A_{22} for (b) $\sigma=0.1/\sqrt{3}$ ($\max\langle F^2\rangle=54$) and (c) $\sigma=0.5/\sqrt{3}$ ($\max\langle F^2\rangle=250$). $\omega=100$ Hz. Again, the range of values is divided into 10 contours from dark to light.

ments in **A**). We chose to optimize λ_{max}, instead of $\langle F^2\rangle$ or $\langle d^2\rangle$, so that we could characterize the reliability of these control strategies for different noise levels as we increase σ. Otherwise, we would have to find a different control strategy for each noise level. We first optimize λ_{max} with a conjugate gradient method,[21] taking as seed the matrix with the single non-null element $A_{11}=\alpha_{min}$ obtained above. Then, the following optimum control matrix is found:

$$\mathbf{A}_c=\begin{pmatrix} -0.27903 & -0.02445 & 0.04248\\ 0.02551 & -0.46523 & 0.18590\\ 0.04679 & 0.02087 & -0.05430\end{pmatrix},$$

with an exponent $\lambda_{max}=-0.720$. As expected, this improves on the single element control matrix strategy, where the optimal value $\lambda_{max}=-0.4$ was obtained, for $\alpha=\alpha_{min}=-1.11$. Given that a lower Lyapunov exponent has been found, we could expect a considerable improvement of the average distance as a function of the noise level σ, but this is not necessarily the case, as shown in Fig. 3(a). Still, it has an average control effort $\langle F^2\rangle$ much lower that the single element matrix strategy, as seen in Fig. 3(b).

We now notice that, although the optimization of λ_{max} with respect to the control matrix **A** should intuitively work, the fact is that the parameter space in $d\times d$ dimensions is probably complicated, and using a conjugate gradient method with a given seed will at most lead us to a local minimum. This suggests that a global optimizer should be used instead. We choose a Monte Carlo method,[21] which leads to the optimum control matrix

$$\mathbf{A}_m=\begin{pmatrix} -33.65913 & 1.06570 & -13.79109\\ 0.12775 & -31.07222 & -5.77190\\ 20.29251 & -4.29226 & -21.10780\end{pmatrix},$$

giving $\lambda_{max}=-30.3$. This is indeed lower than the minimum found by the previous local method.

In Figs. 3(a) and 3(b), we show the average distance $\langle d^2\rangle$ to the UPO and the average control effort $\langle F^2\rangle$ required to keep the system under control with the matrix $\mathbf{A}_m$. We notice that control has improved considerably with respect to the previous strategies, and that it is much more robust as noise is increased (that is, as σ increases). Thus, the possibility to use the full control matrix, and an adequate choice of the optimization technique, are very relevant in the presence of noise, which may be unavoidable in experimental situations. In particular, the decision to search for a global minimum improved results noticeably, which suggests that more advance techniques for global optimization, such as genetic algorithms[25] or configurational space annealing,[26] would lead to better control strategies. Furthermore, the existence of multiple minima is an interesting and unexpected result in itself, which can have relevant implications in experimental situations. We plan to analyze these issues elsewhere.

B. Higher dimensions

The above method is also applicable to hyperchaotic systems (two or more positive Lyapunov exponents) such as the following system:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix}=\begin{bmatrix}-x_2-x_3\\ x_1+\delta_1 x_2+x_4\\ \delta_2+x_1x_3\\ -\delta_3x_3+\delta_4x_4\end{bmatrix}+\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\\ \xi_4(t)\end{bmatrix}.$$

The variables $\xi_i(t)$ represent a particular realization of a noise sequence. Let us assume first that $\xi_i(t)=0$. For the initial conditions $(x_{1,0},x_{2,0},x_{3,0},x_{4,0})=(-10,-6,0,10)$, with

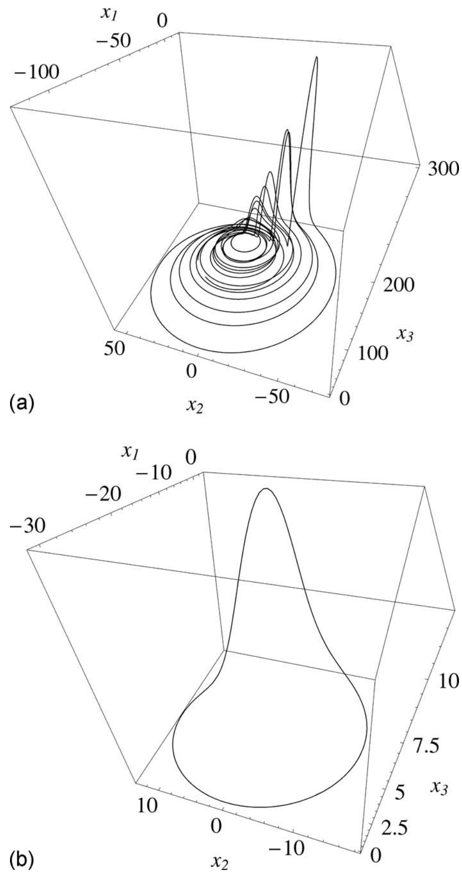

FIG. 7. (a) The hyperchaotic attractor for the parameters $\delta_1=0.25$, $\delta_2=3$, $\delta_3=0.5$, and $\delta_4=0.05$ showing a chaotic behavior. (b) A UPO in the hyperchaotic system. Both figures correspond to a projection onto the space $x_1x_2x_3$.

$\delta_1=0.25$, $\delta_2=3$, $\delta_3=0.5$, and $\delta_4=0.05$, it generates the orbit shown in Fig. 7(a) (projected onto the $x_1x_2x_3$ space). One UPO is obtained from the initial condition $(x_{1,\mathrm{UPO}}, x_{2,\mathrm{UPO}}, x_{3,\mathrm{UPO}}, x_{4,\mathrm{UPO}}) \approx (-24.9, -12.6, 0.117, 16.0)$ with period $\tau=6.19$, $\mathrm{Re}(\lambda_1)=0.18$ and $\mathrm{Re}(\lambda_2)=0.02$. The result is shown in Fig. 7(b).

For this hyperchaotic case, where we have two positive Lyapunov exponents, we rapidly realize that we need at least two nonzero parameters in the control matrix to achieve control. If we are interested in an optimal control scheme, we may need to resort to finding the full matrix **A** that minimizes $\lambda_{\max}$. The Monte Carlo procedure yields the solution

$$\mathbf{A}_H = \begin{pmatrix} -9.91786 & -0.80592 & -2.23062 & 0.93314 \\ 1.39533 & -9.11671 & 0.05988 & -1.12702 \\ -0.18147 & 0.02410 & -1.57179 & -0.35275 \\ -1.44968 & -1.35083 & -2.11769 & -7.84050 \end{pmatrix},$$

with $\lambda_{\max}=-8.59$. With this value for the maximum exponent, we expect that the controlled UPO, shown in Fig. 8, can endure a fairly strong noise. In this case, we used $\sigma=10/\sqrt{3}$ and $\omega=100$ Hz.

We note that our optimal control strategy, using the full matrix $\mathbf{A}_H$, permits a considerable degree of stability of the controlled system, even when strong noise is applied. In Fig. 9, we show the control effort $\langle F^2\rangle$ required to keep the system under control as a function of σ. In this case, the slope of $\langle F^2\rangle$ with respect to σ changes as σ is increased in an almost quadratic fashion as expected from our analysis above and in the Appendix.

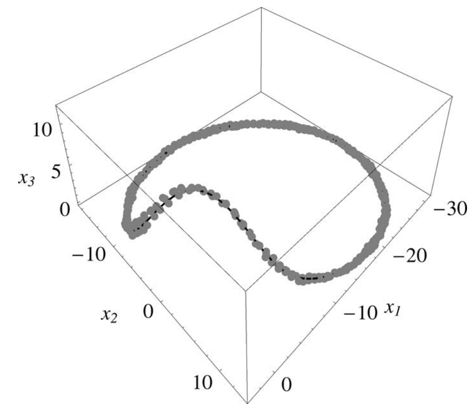

FIG. 8. UPO in the hyperchaotic attractor for the parameters $a=0.25$, $b=3$, $c=0.5$, and $d=0.05$. Projection on the space $x_1x_2x_3$. Gray dots represent a controlled trajectory with added noise. Note the high stability even for this strong noise, namely $\sigma=10/\sqrt{3}$ and $\omega=100$ Hz.

IV. HANDLING NOISE

An issue usually not addressed by other authors is how to control systems that have inherent noise, from estimating the UPO, then computing the exponents, and to finally implementing the optimization routine and constructing an optimal matrix **A**. These issues may be of particular relevance in experimental situations. We will demonstrate how to use the approach described in this paper to control a Rossler system with intrinsic noise, $\sigma=0.1/\sqrt{3}$ and $\omega=100$ Hz.

In a noisy system, the estimation of the UPO may not be a trivial task, since the same initial condition would in general produce many different trajectories $x_i^k(t)$ (here, $i=1,\ldots,d$ labels the space dimensions, and $k=1,\ldots,N$ labels trajectories resulting from the same initial condition evolving under N different noise sequences). Thus, finding the minimum of $H(\vec{x}_0,\tau)$ has to be understood in an average sense, i.e., $\vec{x}_i(\tau)=\Sigma_k\vec{x}_i^k(\tau)/N$. Another method that may be more suitable for large noise levels is to resort to singular value

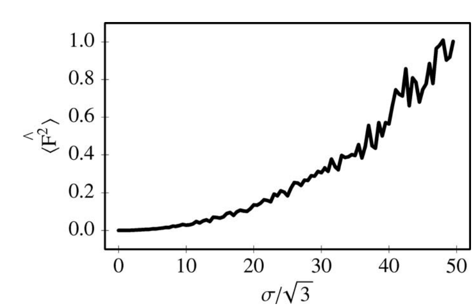

FIG. 9. The average control effort $\langle F^2\rangle$ required to keep the hyperchaotic system under control as a function of σ with the control matrix $\mathbf{A}_H$, using $\omega=100$ Hz. $\langle F^2\rangle$ has been normalized to its maximum value. The close to quadratic dependence with σ is expected from the analysis described above and in the Appendix.

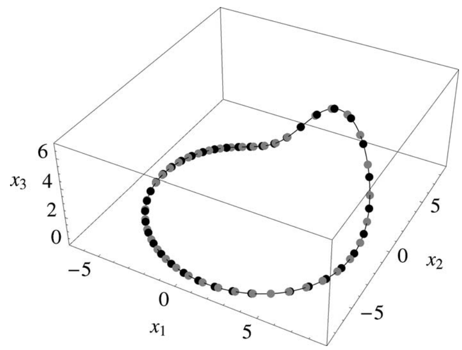

FIG. 10. The estimated UPO using the two noise reduction methods outlined in Sec. IV (gray and black dots) and the one calculated without noise (continuous line). The three curves are very similar.

decomposition (SVD), which is generally used in signal processing of images.[21] We start by constructing the matrix

$$M^i_{k,j} = x^k_i(j\Delta t),$$

where $\Delta t = \tau/N_t$ for some integer N_t. The SVD transformation of $\mathbf{M} = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^T$ is described in general books on linear algebra or in Ref. 21. The first column of $\mathbf{U}$ should be proportional to the cleaned orbit, which can be rescaled by $x_i(0)$. Then we repeat this analysis for $i=1,\ldots,d$, from which we can obtain the value of $H(\vec{x}_0,\tau)$. For this approach to give sensible results, we need to be close to a UPO for obvious reasons. Then we search for a local minimum of $H(\vec{x}_0,\tau)$ over $\vec{x}_0$ and τ as in Sec. III. The result of the two noise cleaning procedures outlined, and the UPO estimated without noise, are shown in Fig. 10. We observe that the trajectories are very similar. For the rest of the paper, we use the UPO $\vec{x}^*_S(t)$ estimated by the SVD procedure. In general, we prefer to use the SVD method as a cleaning strategy, because of its orthogonal properties, and because it gives us a consistent method to correct for the noise level through the singular values.[21]

The second step is to compute the Lyapunov exponents. Since noise is present, standard estimation procedures will give considerable fluctuations. Rather, we estimate finite exponents from Eq. (2), that is, we integrate numerically an initially small, not necessarily infinitesimal, perturbation $\vec{x}(t) = \vec{x}^*_S(t) + \vec{v}^{(k)}(t)$ from $t=0 \to \tau$. In essence, this analysis simulates an experimental situation in which we do not necessarily have a handle on the exact dynamical equations. Notice that in practice we can do this calculation in real time, since we only need to measure $\vec{x}(t)$ and $\vec{x}(t+\tau)$ every time these two states are close to the UPO.

We start all the trajectories with $|\vec{v}^{(k)}(0)| = \delta_0$. We will resort again to singular value decomposition to clean the exponents. In order to compute the matrix $\mathbf{B}(\tau) = \mathbf{V}(\tau)\mathbf{V}^{-1}(0)$, we need to invert the matrix $\mathbf{V}(0)$. We can take a nonsquare matrix of initial conditions

$$\mathbf{V}(0) = \{\vec{v}_1(t), \ldots, \vec{v}_N(t)\},$$

with $N \geq d$, with d the dimension of the system. These N initial conditions can be chosen at random, or taken from the dynamics of the system each time the trajectory passes close to the UPO, in the case of experiments. Even though $\mathbf{V}(0)$ is

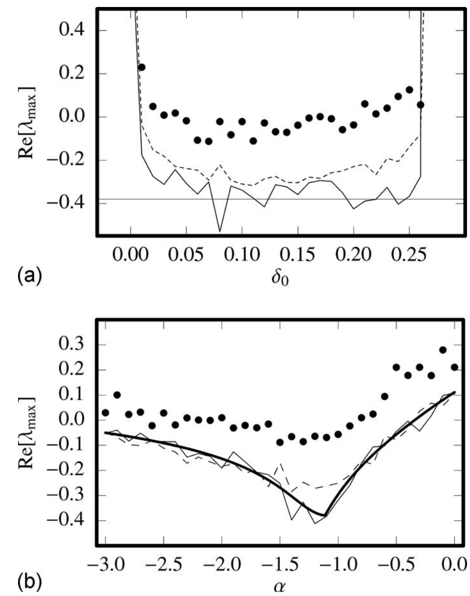

FIG. 11. Maximum exponent λ_{max} calculated for a generalized Rossler system with intrinsic noise. (a) The value of λ_{max} at $\alpha_R = -1.1$ as a function of δ_0 using the average of 40 sets of $N=3$ initial conditions (dotted line), the average of 12 sets of $N=10$ initial conditions (dashed line), and the result of 1 set of $N=120$ initial conditions (thin line). The estimated value of λ_{max} using the infinitesimal approach is shown as the horizontal thick line. (b) The estimated maximum exponent as a function of α using the SVD cleaning procedure for the same sets as before. We take $\sigma = 0.1/\sqrt{3}$ with $\omega = 100$ Hz.

a nonsquare matrix, we can compute its pseudoinverse[21] and estimate a square $d \times d$ matrix $\mathbf{B}(\tau) = \mathbf{V}(\tau)\mathbf{V}(0)^{-1}$. For simplicity, we assume again a control strategy based on the matrix with the single non-null element $A_{11} = \alpha$. Of course we could resort to the full matrix $\mathbf{A}$ if necessary, i.e., if the noise level is larger than what we assumed for this demonstration.

Let us take 120 initial trajectories close to $\vec{x}^*_S(0)$ for $\alpha_R = -1.1$, and construct 40 different sets of $N=3$ trajectories with $|\vec{v}^{(k)}(0)| = \delta_0$, $k=1,\ldots,120$. For each set we compute λ_{max}, and average it over the 40 sets. Figure 11(a) shows this estimated value of λ_{max} as a function of δ_0. We can then repeat the analysis for 12 sets of $N=10$, and for 1 set of $N=120$, using the same 120 vectors for comparison. We see that the set of $N=120$ is numerically more stable for this noise level. Also notice that the behavior illustrated in Fig. 11(a) is expected. For small δ_0 the noise becomes more relevant than the dynamics, and the distance to $\vec{x}^*_S(t)$ should increase in time. For large δ_0, we start sampling other regions of phase space, and not the local properties of the UPO in question, so the distance should also be large. Therefore, there is an optimal range in which estimating λ_{max} makes sense. With this information, we choose $\delta_0 = 0.1$ and we now proceed to estimate λ_{max} using our cleaning procedure as a function of α, as shown in Fig. 11(b) for each set of trajectories.

We take the case of $\alpha_R = -1.1$, as suggested by our cleaning procedure of Fig. 11, that gave $\lambda_{max} \approx -0.4$ using one set of $N=120$ vectors. With this value for the maximum exponent, the very unstable orbit of Fig. 12 was controlled directly from the noisy equations, using the control strategy $\mathbf{A} = \alpha_R \mathbf{J}^{11}$.

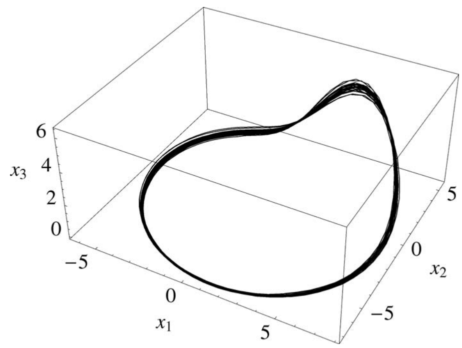

FIG. 12. The controlled UPO of the Rossler system with $\alpha_R = -1.1$ using the control strategy with cleaning. The transient has been removed.

Incidentally, this SVD procedure seems to be a proper manner to estimate Lyapunov exponents, finite or infinitesimal, since it consistently reduces the observed fluctuations in exponent estimation, especially in the exponents with lower real parts as the number of components increases.

V. CONCLUSIONS

We have described a simple method to estimate the Lyapunov spectrum and control a known unstable periodic orbit (UPO) in the presence of noise, approaching the issue of controlling the UPO as an optimization problem. The idea is illustrated with the Rossler system, the Lorenz system, and a hyperchaotic system that has two exponents with positive real parts. For a given system, an optimal control matrix can be calculated, in the sense of minimizing either the maximum of the real parts of the Lyapunov exponents $\lambda_{\max}$, the average distance to the UPO $\langle d^2 \rangle$, the average effort required to control the system $\langle F^2 \rangle$, or other possible criteria, depending on the experimental setup. For small noise levels, the optimal average effort $\langle F^2 \rangle$ required to control the system seems to occur close to the condition for $\lambda_{\max}=0$, and control can be achieved with a single-entry control matrix $A_{11}=\alpha$. As the noise level is increased, optimal effort seems to occur for more negative values of α. When there is strong inherent noise in the system, we saw that the use of the full matrix $\mathbf{A}$ may be required to achieve control. These optimal matrices were found using standard optimization methods such as conjugate gradient or Monte Carlo methods. It is expected that these methods yield better control than the single-entry case, as they explore a larger set of control matrices, and indeed lower values of $\lambda_{\max}$ were found. The strategy obtained with the Monte Carlo method, which gave a very negative value for $\lambda_{\max}$, allowed us to confront a very large level of noise as suggested by $\langle F^2 \rangle$ and $\langle d^2 \rangle$ in Fig. 3. Eventually, it could be more appropriate to use more advanced optimization techniques, such as genetic algorithms[25] or configurational space annealing.[26]

Particular attention was given to the problem of handling noise that can affect considerably the estimation of the UPO itself and the exponents, hence a cleaning strategy based on singular value decomposition (SVD) was developed. In general, it is preferable to use the SVD method as a cleaning strategy, because of its orthogonal properties, and because it gives us a consistent method to correct for the noise level through the singular values.[21] This strategy establishes a consistent way to approach noisy systems. In particular, it may be relevant in higher dimensions and experimental situations, and can be easily adapted as a parametric control strategy.

Another important issue related to noise is the appearance of spurious attractors. These are not UPO in the native system, as far as we could resolve, but have a similar period τ to the desired trajectory, and arise due to a nontrivial interaction between the native system and the control procedure. It has been proposed to replace the UPO in the control signal by a state vector delayed in τ, a scheme that may reduce complexity in an experimental realization. However, in light of our results, it may converge to spurious UPOs instead of the desired trajectory. As it may not be possible, in general, to distinguish a UPO induced by the specific control strategy from a native one, based only on the knowledge of the period τ, this is an issue that deserves attention when controlling noisy systems.

Finally, we can mention that this procedure of finding the optimal matrix $\mathbf{A}$ can be combined with the UPO search routine for cases in which it may be difficult to estimate a UPO, or in cases in which we may not be interested in a particular UPO (see Ref. 18 for a related approach in maps).

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APPENDIX: ANALYTICAL ESTIMATION OF THE CONTROL EFFORT

Let us analyze what is the dependence of $\langle F^2 \rangle$ as we change $\mathbf{A}$ and the noise level. Given the Floquet solutions $\vec{\eta}_i(t)=\vec{\zeta}_i(t)e^{\lambda_i t}$ of the noiseless perturbation equations, we introduce the vector $\vec{\xi}(t)=(\xi_1(t)\xi_2(t)\dots,\xi_d(t))$, which describes a particular noise sequence, and write in vectorial form the general perturbation in the presence of noise as

$$\vec{V}(t)=\boldsymbol{\zeta}(t)\mathbf{E}(t)\vec{a}(t)=\sum a_i(t)\,\vec{\eta}_i(t)=\sum a_i(t)\vec{\zeta}_i(t)e^{\lambda_i t},$$

with the definitions for the basis matrix $\boldsymbol{\zeta}(t) =(\vec{\zeta}_1(t)\vec{\zeta}_2(t)\dots,\vec{\zeta}_d(t))$ and

$$\mathbf{E}(t)=\begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_d t} \end{pmatrix}.$$

The vector $\vec{a}(t)$ contains the information about the initial condition and particular noise sequence. The solution to the noisy equations can now be written as

$$\vec{V}(t) = \boldsymbol{\zeta}(t)\mathbf{E}(t)\vec{a}(0) + \boldsymbol{\zeta}(t)\int_0^t \mathbf{E}(t-s)\boldsymbol{\zeta}^{-1}(s)\vec{\xi}(s)ds,$$

where the average effort is given by

$$\langle F^2\rangle = \frac{1}{T}\int_0^T \langle \vec{V}(t)^\dagger |\mathbf{A}|^2 \vec{V}(t)\rangle dt.$$

We first note that when we introduce $\vec{V}(t)$ in the above expression, and take the limit $T\to\infty$, the initial conditions do not contribute if $\lambda_{max}<0$. Hence, we need to analyze

$$\int_0^t\int_0^t ds_1 ds_2 \langle \vec{\xi}^\dagger(s_1)\boldsymbol{\zeta}^{-\dagger}(s_1)\mathbf{E}^\dagger(t-s_1)\boldsymbol{\zeta}^\dagger(t)|\mathbf{A}|^2\rangle \cdot \langle \boldsymbol{\zeta}(t)\mathbf{E}(t-s_2)\boldsymbol{\zeta}^{-1}(s_2)\vec{\xi}(s_2)\rangle.$$

Given that

$$\langle \xi_i^\dagger(s_1)\xi_j(s_2)\rangle = \sigma^2\delta(s_1-s_2)\delta_{ij},$$

we obtain

$$\langle F^2\rangle \propto \sigma^2\int_0^t ds\sum_i [\boldsymbol{\zeta}^{-\dagger}(s)\mathbf{E}^\dagger(t-s)\mathbf{Q}(t)\mathbf{E}(t-s)\boldsymbol{\zeta}^{-1}(s)]_{ii},$$

where $\mathbf{Q}(t)=\boldsymbol{\zeta}^\dagger(t)|\mathbf{A}|^2\boldsymbol{\zeta}(t)$. If we assume the control strategy that uses a matrix with the single non-null element $A_{11}=\alpha$, we get $Q_{ij}=\alpha^2\zeta_{1i}^*\zeta_{1j}$. Hence, the noise makes the different directions interact, i.e., the problem becomes a tensorial one, as expected for dynamical noise. Still it is interesting to note the quadratic dependence on σ. Of course, the matrices inside the integral depend on α as the basis $\boldsymbol{\zeta}(t)$ and the exponents $\mathbf{E}(t)$ change with the control parameter.

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